A Note on Solutions of the Korteweg-de Vries Equation

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Z. Naturforsch. **40 a,** 193–194 (1985); received October 22, 1984

A new method to construct solutions of the Korteweg-de Vries equation and an example of such a solution are given.

In recent years, a number of interesting papers have appeared on the Korteweg-de Vries (KdV) equation solved by various exact techniques [1–22]. In spite of this, the main purpose of this note is to present a new way of finding solutions of the KdV equation. We write

$$p_t + \beta p p_x + \alpha p_{xxx} = 0\,, \tag{1}$$

where β and α are nonlinearity and dispersion coefficients, respectively. The subscripts x and t indicate partial differentiations.

An introducing the function w, which has been done for the first time by Wahlquist and Estabrook [1],

$$p \equiv -w_x\,, \tag{2}$$

and substituting it in (1) we obtain

$$w_t = \frac{\beta}{2} w_x^2 - \alpha w_{xxx} \equiv \frac{\beta}{2} p^2 + \alpha p_{xx}, + F\,, \quad (F = \text{const}) \tag{3}$$

where the partial derivatives of w are chosen to be

$$w_x = g\,(w^2 - k^2) \tag{4}$$

$$w_t = e\,(w^2 - f k^2)\,. \tag{5}$$

Keeping $F = 0$ let us transform (4) and (5) into the form

$$w_t - e/g\; w_x = 0\,, \tag{7}$$

$$w_x = g\,(w^2 - k^2)\,. \tag{8}$$

We immediately obtain from (7)

$$w = w\,(e/g\;t + x) \equiv w\,(\eta)\,. \tag{9}$$

Thus (8) takes the form of Riccati equation

$$w_\eta = g\,(w^2 - k^2)\,. \tag{10}$$

Its solutions are connected via the ladder

$$w_{i+1} = w_i + 1/z_{i+1}\,, \quad i = 0, 1, 2, \ldots, \tag{11}$$

where

$$z_{(i+1)\eta} + 2\,g\,w_i\,z_{i+1} = -g\,. \tag{12}$$

Choosing $w_0 = k$ we obtain

$$w_1 = k + \frac{2\,k}{2\,k\,C\exp(-2\,g\,k\,\eta) - 1}\,, \quad C = \text{const}, \tag{13}$$

and then from (2)

$$p_1 = -\frac{2\,k^3\,g\,C}{\left[\dfrac{2\,k\,C\exp(-k\,g\,\eta) - \exp(k\,g\,\eta)}{2}\right]^2}\,. \tag{14}$$

In case when $C = -1/2\,k$ or $C = 1/2\,k$ we have soliton,

$$p_1 = g\,k^2\,\text{sech}^2\,g\,k\,\eta \tag{15}$$

or shock wave [1],

$$p_1 = -g\,k^2\,\text{csch}^2\,g\,k\,\eta\,. \tag{16}$$

Taking (12) into account and using (14), from (11) we obtain again

$$w_2 = k + \frac{2\,k}{2\,k\,C\exp(-2\,g\,k\,\eta) - 1} + \frac{4\,C\,k^2\exp(-2\,g\,k\,\eta)}{[4\,C C_1\,k^2 - \ln|1 - 2\,k\,C\exp(-2\,g\,k\,\eta)|]\ln|1 - 2\,k\,C\exp(-2\,g\,k\,\eta)|}\,,$$

$$C_1 = \text{const}. \tag{17}$$

It is a new solution of the KdV equation.

To end this note we would like to say that we can also obtain from (11) other solutions of the KdV equation having described the function z_{i+1}. Unfortunately they cannot be expressed with the help of analytical functions in general.

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k is a spectral parameter.

It is easy to note that

$$g = \beta/12\,\alpha\,, \quad e = -\beta\,g^2\,k^2/3\,, \qquad f = 1 - \frac{1}{e\,k^2}\,F\,. \tag{6}$$

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Verantwortlich für den Inhalt: A. KLEMM
Satz und Druck: Konrad Triltsch, Würzburg